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FINITENESS PROPERTIES OF CERTAIN ARITHMETIC GROUPS IN THE FUNCTION FIELD CASE

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ABSTRACT

It is proved that the finiteness length of $\Gamma = \mathrm{SL}_n(\mathbb{F}_q[t])$ is $n-2$ if $n \geq 2$ and $q \geq 2^{n-2}$. The proof consists in studying the homotopy type of a certain Γ-invariant filtration of an appropriate Bruhat–Tits building on which Γ acts.

Introduction

Recall that a group Γ is said to be of type FP_n if the $\mathbb{Z}\Gamma$-model $\mathbb{Z}$ admits a projective resolution which is finitely generated in dimensions $\leq n$, see [23]. For example, Γ is of type FP_1 if and only if Γ is finitely generated, and Γ is of type FP_2 if it is finitely presented. The converse of the last assertion is not known. So one defines a group to be of type $F_n, n \geq 2$, if it is finitely presented and of type FP_n. A group Γ is of type F_n iff there is an Eilenberg–MacLane complex $K(\Gamma, 1)$ with finite n-skeleton ([31, 32], cf. [13]). For a given group Γ let $\phi(\Gamma)$ be the largest n $(0 \leq n \leq \infty)$ such that Γ is of type F_n. We call $\phi(\Gamma)$ the **finiteness length** of Γ.

The first example of a group of type F_2 not of type F_3 was given by Stallings [27], and Bieri [8] extended this to a sequence of groups Γ_n of finiteness length n. Further such examples can be found in [3, 4, 14, 28]. In this paper we give another such sequence

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THEOREM: *$\phi(\mathrm{SL}_n(\mathbb{F}_q[t])) = n-2$ if $n \geq 2$ and $q \geq 2^{n-2}$.*

Here $\mathbb{F}_q$ is the finite field with q elements.

We actually show that for $n \geq 2$ and $q \geq 2^{n-2}$ the group $\mathrm{SL}_n(\mathbb{F}_q[t])$ is not of type FP_{n-1} over any ring R with $1 \neq 0$ (Theorem 1.6).

Let us recall what is known about the finiteness properties of arithmetic groups (for more details and references cf. [15]). In case of characteristic zero, arithmetic groups in general and S-arithmetic subgroups of reductive groups enjoy all finiteness properties, in particular they have infinite finiteness length [21, 11, 12]. For S-arithmetic subgroups of solvable groups there is only a partial understanding of the finiteness properties; see [1, 3, 4, 9, 14].

In case of positive characteristic the situation is much less understood. Let K be a function field of transcendence degree one over the finite field $k = \mathbb{F}_q$, let S be a finite set of places of K and let o_S be the ring of those functions in K which are holomorphic outside of S. Let G be a reductive linear algebraic group over K and let $\Gamma = G(o_S)$ be the corresponding S-arithmetic group. There is a complete answer to the question when Γ is finitely generated and a conjecture as to when Γ is finitely presented, verified in several cases (see [7]). In particular, the following cases of our result were known: $\mathrm{SL}_2(\mathbb{F}_q[t])$ is not finitely generated ([20], cf. [24]), $\mathrm{SL}_3(\mathbb{F}_q[t])$ is finitely generated but not FP_2([6]) and $\mathrm{SL}_n(\mathbb{F}_q[t])$ is finitely presented for $n \geq 4$ ([22]) (we obtain this only for $q \geq 2^{n-2}$).

The finiteness length ≥ 3 of S-arithmetic groups Γ for function fields had been computed in two cases only: In the cocompact case — i.e., the K-rank of G is zero — one has $\phi(\Gamma) = \infty$ ([17, 23]). Stuhler [28] proved

$$\phi(\mathrm{SL}_2(o_S)) = \mathrm{card}\, S - 1.$$

The main theorem of this paper has also been obtained independently by Abramenko [5] with a better bound for q, namely $q \geq \max_i \binom{n-2}{i}$. It is not clear if the assumption about q is necessary. The first group excluded, namely $\mathrm{SL}_5(\mathbb{F}_2[t])$, has been shown by Abramenko [5] to have finiteness length 3.

The present proof may be of interest because it is short and easy if one admits certain facts about buildings and most of it holds for Chevalley groups instead of SL_n (see 4.3). My original proof, as announced in a talk at the Oberwolfach conference on topological methods in group theory in June 1986, involved a rather explicit description of the gallery distance in the Bruhat–Tits building of GL_n.

Here is an outline of the proof and of the contents of the paper. We let $\Gamma = \mathrm{SL}_n(\mathbb{F}_q[t])$ act on the Bruhat–Tits building X of $V = K^n, K = \mathbb{F}_q(t)$, and filter X as follows. Let X_d be the subcomplex of simplices having gallery distance $\leq d$ from the Γ-orbit of a certain chamber C_0. The main point is to verify that X_{d+1} can be obtained from X_d by attaching $(n-1)$-cells up to homotopy. The theorem then follows from a result of K. Brown.

In section 1 we define the filtration and show how the theorem follows once we know that X_{d+1} is obtained from X_d by attaching $(n-1)$-cells up to homotopy. In section 2 we define the Γ-restriction of a chamber in X_{d+1}, not in X_d. The images of these Γ-restrictions will parameterize sets of cells to be attached (see 4.2). In section 3 we describe a fundamental set for Γ in X. In section 4 we see that it suffices to prove that a certain complex $T(\rho)$ is spherical. In section 5 we see that $T(\rho)$ is isomorphic to a certain subcomplex of a Tits (sic! not Bruhat–Tits) building, hence spherical by the main result of [2] — which was proved with just this application in mind.

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1. The Bruhat–Tits Building and its Filtration

We first recall the definition of the Bruhat–Tits building for $\mathrm{GL}(V), V$ a vector space over a field with discrete valuation. We establish notations to describe our filtration. We then state the main result concerning this filtration. We finally show how it implies the result about the finiteness length of $\mathrm{SL}_{n+1}(\mathbb{F}_q[t])$. Note that the dimension of our vector space V is always $n+1$, except in the introduction, where it was n.

1.1 The Bruhat–Tits Building of GL_{n+1}. Let K be a field with discrete valuation v, let R be the corresponding valuation ring $R = \{x \in K | v(x) \geq 0\}$. R has a unique maximal ideal πR, where π is any element with $v(\pi) = 1$. Let V be a vector space of dimension $n+1$ over K. A **lattice** L in V is a finitely generated R-submodule of V which generates V as a vector space. The set of lattices in V is a poset (= partially ordered set) with inclusion as order relation. Two lattices L, L' in V are called **equivalent** if there is an element $\alpha \in K^*$ such that $L' = \alpha L$. Let $\Lambda = [L]$ be the equivalence class of the lattice L. We shall consider the simplicial complex X whose vertices are the classes of lattices in V and whose simplices σ are those non-empty sets of classes of lattices for which $\{L | [L] \in \sigma\}$ is a chain. So σ is a simplex iff σ can be written in the form $\sigma = \{[L_0], \dots, [L_s]\}$ such that $\pi L_s < L_0 < \cdots < L_s$. So X is the Bruhat–Tits building of the group $\mathrm{GL}(V)$ over the field K with discrete valuation v. The group $\mathrm{GL}(V)$ acts simplicially on X.

1.2 The Gallery Distance. Recall the following notions concerning buildings. The simplices of maximal dimension in a building Δ are called **chambers**. Two chambers C, C' are called **adjacent** if $C = C'$ or $C \cap C'$ is of codimension one in both C and C'. A sequence $C_0 = C, C_1, \dots, C_r = C'$ of chambers is called a **gallery** from C to C' of length r if C_{i-1} and C_i are adjacent for every $i = 1, \dots, r$. The **(gallery) distance** $d(C, C')$ of two chambers C, C' is defined as the minimum of the lengths of galleries from C to C'. It is defined because any two chambers of a building can be connected by a gallery. Since any simplex of a building is contained in a chamber, it makes sense to define the distance $d(\sigma, \sigma')$ of two simplices σ, σ' of a building as the minimum of distances $d(C, C')$ of chambers $C \supset \sigma, C' \supset \sigma'$. Note that this is not a metric: $d(\sigma, \sigma') = 0$ iff there is a chamber containing $\sigma \cup \sigma'$; and $d(\sigma, \sigma'') \leq d(\sigma, \sigma') + d(\sigma', \sigma'')$ does not hold in general, e.g. the two summands on the right may be zero while $d(\sigma, \sigma'') \neq 0$.

1.3 The Filtration. Let now $K = k(t)$ be the field of rational functions in one variable t over the field k. Let $v = v_\infty$ be the valuation given by the order of a function at ∞, i.e.

$$v\left(\frac{f}{g}\right) = \deg g - \deg f, \qquad f, g \in k[t], \quad g \neq 0.$$

Let $V = K^{n+1}$ with its standard basis $e_0, \dots, e_n$. Since $\mathrm{GL}(V) = \mathrm{GL}_{n+1}(K)$ acts on X, so does its subgroup

$$\Gamma = \mathrm{SL}_{n+1}(k[t]).$$

Let C_0 be the chamber of X whose vertices are the classes of lattices $L(m) = \oplus_{i=0}^{n} \pi^{-m_i} R e_i$, where $m = (m_0, \dots, m_n) \in \mathbf{Z}^{n+1}, m_0 \geq m_1 \geq \cdots \geq m_n$ and $0 \leq m_i \leq 1$. Let X_d be the subcomplex of X of simplices of distance $\leq d$ from some chamber $\gamma C_0, \gamma \in \Gamma$.

1.4 THEOREM: *X_{d+1} is obtained from X_d by attaching n-cells, up to homotopy, if $q = \text{card } k \geq 2^{n-1}$.*

Note that k need not be finite.

The proof of 1.4 will be given in the next sections. The theorem about the finiteness length of Γ follows from 1.4 and the following special case of a result of K. Brown [14, Corollary 3.3].

1.5 THEOREM: *Let X be a contractible CW-complex on which a group Γ acts cellularly. Suppose the stabilizer of every cell is finite. Let $X_d, d \geq 1$, be an infinite filtration of X by Γ-invariant subcomplexes such that each X_d is finite modulo Γ. Suppose X_{d+1} is obtained from X_d by attaching n-cells up to homotopy for all sufficiently large d. Then Γ is of type F_{n-1} but not of type FP_n over any ring with $1 \neq 0$.*

1.6 THEOREM: *Let $n \geq 2$ and $\mathbb{F}_q$ be a finite field with at least 2^{n-2} elements. Then $\Gamma = \mathrm{SL}_n(\mathbb{F}_q[t])$ is of type F_{n-2}, but Γ is not of type FP_{n-1} over any ring with $1 \neq 0$.*

Proof: Note that the theorem is stated for SL_n, whereas the proof uses SL_{n+1}. Recall that the Bruhat–Tits building is contractible [25]. For every vertex Λ of X the stabilizer $\mathrm{SL}_{n+1,\Lambda}$ of Λ is conjugate to the stabilizer of a vertex of C_0, hence all the entries of $\mathrm{SL}_{n+1,\Lambda}$ are bounded below with respect to the valuation v, so Γ_Λ is finite if k is finite. ∎

2. The Γ-Restriction

In this section we define the Γ-restriction $\mathcal{R}^\Gamma(C)$ of a chamber C. The images of the Γ-restrictions will parametrize the n-cells we shall attach to X_d to obtain X_{d+1} (see 4.2). In this section we only suppose that a group Γ acts simplicially on a building Δ such that the action admits a fundamental domain. It is a basic fact that the action of Γ on X as in section 1 does admit a fundamental domain (see 3.3).

2.1 THE RESTRICTION. In this subsection we define a notion of restriction, very closely related to the notion of restriction of a shelling (cf. [10]). We do not take the action of Γ into account yet.

Let Δ be an arbitrary building. Fix a chamber C_0 of Δ. For any integer $d \geq 0$ let $\Delta_d(C_0)$ be the set of simplices of Δ of distance $\leq d$ from C_0. For a simplex A of a simplicial complex Δ let $\overline{A}$ denote the subcomplex of simplices contained in A. Then for any chamber C of our building Δ such that $d(C_0, C) = d + 1$ we have

$$\bar{C} \cap \Delta_d(C_0) = \bigcup_{A \in \mathcal{F}} \bar{A},$$

where $\mathcal{F}$ is a set of codimension 1 faces of C ([25], cf. [15] IV. 6 Proposition).

For a fixed chamber C_0, define the restriction of C as

$$\begin{aligned}\mathcal{R}(C) &:= \{\text{vertices } v \text{ of } C \text{ such that } d(C_0, C - \{v\}) \leq d\} \\ &= \bigcup_{A \in \mathcal{F}} C \setminus A\end{aligned}$$

if $d(C_0, C) = d + 1$. Then $\mathcal{R}(C)$ is the smallest simplex $\sigma \subset C$ not contained in $\Delta_d(C_0)$: i.e., $\sigma \subset C$ is in $\Delta_d(C_0)$ iff $\sigma \not\supset \mathcal{R}(C)$.

The name restriction was chosen by analogy with the term restriction used for shellings ([10]).

Remark: One can describe $\mathcal{R}(C)$ in terms of the corresponding Coxeter system (W, S) as follows. For $w \in W$ the **descent set** $\mathcal{D}(w)$ of w is

$$\mathcal{D}(w) = \{s \in S | \ell(ws) < \ell(w)\}.$$

Here $\ell(w)$ is the length of a word representing w with respect to the distinguished set S of generators of W. With notations as above, if $C = wC_0$ for $w \in W$, then $\mathcal{F}$ is the set of codimension one faces $C \cap C'$, where $C' = w'C_0$ and $\ell(w') < \ell(w)$. Hence $x \in \mathcal{R}(C)$ iff the type of x is in $\mathcal{D}(w)$. So $\mathcal{R}(C)$ is the face of C of type $\mathcal{D}(w)$ (cf. [10] Theorem 2.1). ∎

2.2. Let a group Γ act simplicially on a building Δ. Define

$$\Delta_d(\Gamma C_0) = \{\sigma | d(\sigma, \Gamma C_0) \leq d\} = \Gamma \Delta_d(C_0),$$

where of course $d(S, T) = \inf\{d(\sigma, \tau) | \sigma \in S, \tau \in T\}$ for any two sets S, T of simplices of Δ. The filtration X_d of X given in section 1 is of this form, $X_d = \Delta_d(\Gamma C_0), d \in \mathbb{N}$, for a certain chamber C_0. We define the **Γ-restriction** of a chamber C in $\Delta_{d+1}(\Gamma C_0)$ not in $\Delta_d(\Gamma C_0)$:

$$\mathcal{R}^{\Gamma}(C) = \{v \in C | d(\Gamma C_0, C - \{v\}) \leq d\}.$$

2.3. Let a group Γ act simplicially on a simplicial complex Δ. A subcomplex F of Δ is called a **fundamental set** (in the strictest sense) if the orbit $\{\gamma\sigma | \sigma \in \Gamma\}$ of any simplex σ of Δ contains exactly one simplex of F.

2.4 LEMMA: *If there is a fundamental set for Γ in the building Δ then*
(a) *For $C \in \Delta_{d+1}(\Gamma C_0), \notin \Delta_d(\Gamma C_0)$ the Γ-restriction $\mathcal{R}^\Gamma(C)$ is the smallest simplex contained in C not in $\Delta_d(\Gamma C_0)$: so $\sigma \subset C$ is in $\Delta_d(\Gamma C_0)$ iff $\sigma \not\supset \mathcal{R}^\Gamma(C)$.*
(b) *For any two chambers C, C' of $\Delta_{d+1}(\Gamma C_0)$ we have $C \cap C' \in \Delta_d(\Gamma C_0)$ or $\mathcal{R}^\Gamma(C) = \mathcal{R}^\Gamma(C')$ and $C = \gamma C'$ for some $\gamma \in \Gamma$.*

Proof: Corresponding to a fundamental set F there is a unique map $r : \Delta \to F$ such that for every simplex σ of Δ we have $r\sigma = \gamma\sigma$ for some $\gamma \in \Gamma$. Then r is a simplicial retraction. It follows that

$$d(\Gamma\sigma, \Gamma\sigma') = d(r\sigma, r\sigma') \tag{2.5}$$

for any two simplices σ, σ' of Δ. Hence if $C_0 \in F$ — which we may assume — we have

$$\Delta_d(\Gamma C_0) \cap F = \Delta_d(C_0) \cap F,$$

hence we obtain

$$r\mathcal{R}^\Gamma(C) = \mathcal{R}(rC) \tag{2.6}$$

and (a), since $\mathcal{R}(rC)$ is the smallest simplex $\sigma \subset rC$ not in $\Delta_d(C_0)$.

If now $C \cap C'$ is not in $\Delta_d(\Gamma C_0)$ then $r(C \cap C')$ is not in $\Delta_d(C_0)$, so there is a smallest simplex $\sigma \subset r(C \cap C')$ not in $\Delta_d(C_0)$, namely $\sigma\mathcal{R}(rC) = \mathcal{R}(rC')$, hence $\mathcal{R}^\Gamma(C) = \mathcal{R}^\Gamma(C')$ by 2.6 and injectivity of r on $C \cap C'$. Furthermore $rC = rC'$ and hence our last claim, since for every simplex σ there is a unique chamber $D \supset \sigma$ such that $d(C_0, \sigma) = d(C_0, D)$ (the chamber D is the projection of C_0 on σ of [29] 2.30, cf. [15] IV. 6 Lemma 1).

3. An Apartment, The Fundamental Set

We now return to the situation of section 1. We describe a subcomplex of X, an "apartment" in the terminology of the theory of buildings. It contains a fundamental set F.

3.1 THE COMPLEX A. Let $\mathbb{Z}$ act on $\mathbb{Z}^{n+1}$ by

$$z + m := m + ze$$

where $e = (1, \ldots, 1) \in \mathbb{Z}^{n+1}$. Let $[m]$ be the orbit of m. Let A be the simplicial complex whose vertices are the $\mathbb{Z}$-orbits in $\mathbb{Z}^{n+1}$ and whose simplices are the non-empty finite subsets τ of $\mathbb{Z}^{n+1}/\mathbb{Z}$ such that $\{m|[m] \in \tau\}$ is a chain in $\mathbb{Z}^{n+1}$. Here the order relation is $m = (m_0, \ldots, m_n) \leq m' = (m'_0, \ldots, m'_n)$ iff $m_i \leq m'_i$ for $i = 0, \ldots, n$.

3.2. Let $e_0, \ldots, e_n$ be the standard basis of $V = K^{n+1}$, as in section 1. For $m \in \mathbb{Z}^{n+1}$ define

$$L(m) = \bigoplus \pi^{-m_i} R e_i$$

and $\Lambda(m) = [L(m)]$. Then the map $A \to X, [m] \to \Lambda(m)$, given an isomorphism of A with the full subcomplex of X containing all the vertices $\Lambda(m), m \in \mathbb{Z}^{n+1}$. The image of A is actually an **apartment** of X (cf. [15] V. 8).

3.3. Let $\mathbb{Z}^{n+1}_{\text{mon}}$ be the set of sequences $m = (m_0, \ldots, m_n)$ in $\mathbb{Z}^{n+1}$ which are monotonically decreasing, i.e., $m_0 \geq m_1 \geq \cdots \geq m_n$. Let F be the full subcomplex of A whose vertices are the orbits $[m], m \in \mathbb{Z}^{n+1}_{\text{mon}}$. It is a basic fact that F is a **fundamental set** for the action of Γ on X ([26], cf. 5.1.5).

4. Attaching Cells

Using the notion of Γ-restriction we parametrize the cells to be attached to X_d to obtain X_{d+1}. We state an explicit version of Theorem 1.4 to be proved in the last section of the paper. It reduces our proof to showing that the complex $T(\rho)$ of 4.1 is spherical.

4.1. Let X and C_0 be as in section 1. In section 2 we defined the notion of Γ-restriction. Let

$$R_{d+1} = \{\mathcal{R}^\Gamma(C) | C \in X_{d+1}, \notin X_d\}.$$

For $\rho \in R_{d+1}$ let

$$S(\rho) = \{\sigma | \sigma \cup \rho \in X_{d+1}\}$$

be the star of ρ in X_{d+1} and

$$T(\rho) = S(\rho) \cap X_d.$$

Here is an explicit version of Theorem 1.4.

4.2 LEMMA:
(a) $X_{d+1} = X_d \cup \bigcup_{\rho \in R_{d+1}} S(\rho)$.
(b) $S(\rho) \cap S(\rho') \subset X_d$ *for* $\rho \neq \rho'$ *in* R_{d+1}.
(c) $S(\rho)$ *is homeomorphic to the cone over* $T(\rho)$.
(d) $T(\rho)$ *is a complex of dimension*
$n-1$ *homotopy equivalent to a wedge of* $(n-1)$*-spheres, if* $q = \#k \geq 2^{n-1}$.

(a) is clear by definition, (b) follows from 2.4(b).

(c) $\sigma \in S(\rho)$ is in $T(\rho)$ iff $\sigma \not\supset \rho$, by 2.4(a). So $T(\rho)$ is the join of $T'(\rho) = \{\sigma \in S(\rho) | \sigma \cap \rho = \emptyset\}$ and the boundary $\partial\rho$ of ρ

$$T(\rho) = \partial\rho * T'(\rho).$$

Hence the cone over $T(\rho)$ is $T'(\rho) * \partial\rho *$ point, homeomorphic to $T'(\rho) * \rho = S(\rho)$.

The proof of (d) will occupy the last section of the paper.

4.3 Remark: Note that the proof so far holds for simple and simply-connected Chevalley groups G over $\mathbb{Z}$ instead of SL_n. The analogue of 4.2(d) is missing: The dimension and homotopy type of $T(\rho)$ has to be determined. ∎

5. The Chamber of Minimal Distance, Transversality or How to Stack Boxes

Given a simplex σ in the fundamental set F of section 3 we give two descriptions of the chamber $C \supset \sigma$ such that $d(C_0, \sigma) = d(C_0, C)$. One involves the concept of transversality to the canonical filtration. The other one describes a certain rule for stacking boxes. The first one implies Proposition 4.2 by a joint result with Abramenko [2].

5.1 THE CANONICAL FILTRATION. We give an elementary description of the canonical filtration [19, 30] in our situation.

Fix an R-lattice L in $V = K^{n+1}$, R the valuation ring for the valuation $v = v_\infty$ of $K = k(t)$, k an arbitrary field, as in 1.3. Let $\bar{k}$ be an algebraic closure of k. For $P \in \bar{k}$ let v_P be the valuation of $\bar{k}(t)$ which assigns to $f \in \bar{k}(t)$ its order at P, i.e., the exponent $\in \mathbb{Z}$ of the factor $(t-P)$ in the decomposition of f into linear factors. Let R_P be the corresponding valuation ring,

$$R_P = \{f \in \bar{k}(t) | v_P(f) \geq 0\} = \bar{k}[t]_{t-P}.$$

Define the degree of $x = (x_0, \ldots, x_n) \in K^{n+1}$ with respect to L as

$$\deg^L(x) = \sum_{P \in \mathbb{P}^1(\bar{k})} \deg^L_P(x),$$

where the sum is taken over $P \in \mathbb{P}^1(\bar{k}) = \bar{k} \cup \{\infty\}$ and, for $P \in \bar{k}$,

$$\begin{aligned}\deg^L_P(x) = \deg_p(x) &= \min\{v_P(x_0), \ldots, v_P(x_n)\} \\ &= \sup\{r \in \mathbb{Z} | (t-P)^{-r} x \in \bigoplus_{i=0}^{n} R_P\, e_i\}\end{aligned}$$

and

$$\deg^L_\infty(x) = \sup\{r \in \mathbb{Z} | \pi^{-r} x \in L\}.$$

Hence

$$\begin{aligned}&\deg^L x = \infty \qquad \text{iff } x = 0,\\ &\deg^L fx = \deg^L x \qquad \text{for } f \in K^*,\\ &\deg^{L(m)} e_i = m_i \qquad \text{for } m = (m_0, \ldots, m_n) \in \mathbb{Z}^{n+1},\\ &\deg^{L(m)} \sum_{j=0}^{i} f_j e_j \le m_i \qquad \text{if } f_i \neq 0 \quad \text{and } m \in \mathbb{Z}^{n+1}_{\text{mon}}.\end{aligned}$$

For $t \in \mathbb{Z}$ define

$$V^t = \text{span}\{x \in K^{n+1} | \deg^L(x) \ge t\}.$$

So for $L = L(m)$, $m \in \mathbb{Z}^{n+1}_{\text{mon}}$, we have

$$V^t = \text{span}\,\{e_i | m_i \ge t\}.$$

The **canonical filtration** $\tau(L)$ corresponding to L is the ascending sequence of different ones among the spaces $V^t, t \in \mathbb{Z}$. So $\tau(L)$ is a flag in V.

Note that τ is Γ-equivariant, i.e., for $\gamma \in \Gamma$ we have

$$\tau(\gamma L) = \gamma \tau(L),$$

since $\deg_P(\gamma x) = \deg_P(x)$ as $\oplus R_P\, e_i$ is Γ-invariant, hence $\deg^{\gamma L}(\gamma x) = \deg^L(x)$.

For every pair $L'' \le L'$ of R-submodules of V a flag τ in V induces a flag $\tau_{L'/L''}$ of R-submodules of L'/L'', namely

$$\tau_{L'/L''} = \{U \cap L' / U \cap L'' | U \in \tau\}.$$

In particular, if $\pi L' \le L'' \le L'$ then L'/L'' is a vector space over the field $R/\pi R = k$ and $\tau_{L'/L''}$ is a flag of subspaces of L'/L''.

Recall that two vector subspaces U_1, U_2 of a vector space W are called **transversal** if $U_1 \cap U_2 = 0$ or $U_1 + U_2 = W$. A vector subspace U of W is called transversal to a set of subspaces of W if U is transversal to every space of this set.

5.1.1 PROPOSITION: *Suppose X, C_0 and F are as in section 1 and section 3.3. Given a simplex σ of F and a chamber $C \supset \sigma$ of X, then $d(C_0, \sigma) = d(C_0, C)$ iff $C \in F$ and for every $[L] \in C$ the following conditions hold. Let L' be the largest lattice $\le L$ such that $[L'] \in \sigma$ and let L'' be the smallest lattice $\ge L$ such that $[L''] \in \sigma$. Then L/L' is transversal to the flag $\tau(L')_{L''/L'}$.*

A proof of 5.1.1 will be given in the next subsection 5.2.

In order to apply the proposition to prove 4.2(d) let us restate the proposition in a way independent of the fundamental set.

Recall that the **Tits building** $T(W)$ of a vector space W is the flag complex of the poset of vector subspaces $U \neq 0, W$ of W ordered by inclusion. Let

$$\sigma = \{[L_0], \dots, [L_s]\}, \qquad L_0 < L_1 < \dots < L_s < \pi^{-1} L_0 =: L_{s+1},$$

be a simplex in X. Define a map λ from the star of σ in X to the complex which is the join of σ and the join of the Tits buildings of the various $L_{i+1}/L_i, i = 0, \dots, s$,

$$\lambda : \operatorname{st}_X \sigma \to \sigma * \mathop{*}_{i=0}^{s} T(L_{i+1}/L_i),$$

by $\lambda([L_i]) = [L_i]$ and $\lambda([L]) = L/L_i$ if $L_i < L < L_{i+1}$. Then λ is an isomorphism of simplicial complexes. For a set $\mathcal{E}$ of vector subspaces of W let $T_{\mathcal{E}}(W)$ be the full subcomplex of $T(W)$ of subspaces $U \neq 0, W$ which are transversal to $\mathcal{E}$.

5.1.2 PROPOSITION: *For a simplex σ in X and a chamber $C \supset \sigma$ in X we have $d(\Gamma C_0, \sigma) = d(\Gamma C_0, C)$ iff*

$$\lambda C \in \sigma * \mathop{*}_{i=0}^{s} T_{\mathcal{E}_i}(L_{i+1}/L_i),$$

where $\mathcal{E}_i = \tau(L_i)_{L_{i+1}/L_i}$.

5.1.1 ⇒ 5.1.2: Let $r : X \to F$ be the retraction onto the fundamental set (see the proof of 2.4). Suppose $rC = \gamma C$ for some $\gamma \in \Gamma$. Then $d(\Gamma C_0, \sigma) = d(\Gamma C_0, C)$ iff $d(C_0, \gamma\sigma) = d(C_0, \gamma C)$ by 2.5 iff

$$\lambda \gamma C \in \gamma\sigma * \mathop{*}_{i=0}^{s} T_{\mathcal{E}_i'}(\gamma L_{i+1}/\gamma L_i)$$

where $\mathcal{E}_i' = \tau(\gamma L_i)_{\gamma L_{i+1}/\gamma L_i}$ by 5.1.1, hence 5.1.2 since everything — including τ — is Γ-equivariant.

5.1.2 ⇒ 4.2(d): If $\mathcal{E}$ is a flag in a vector space W of dimension w, then $T_{\mathcal{E}}(W)$ has dimension $(w-2)$ and is homotopy equivalent to a wedge of $(w-2)$-spheres provided $q = \#k \geq 2^{w-2}$. This is a special case of the main result of [2]. Hence the complex

$$T(\rho) = \partial\rho * T'(\rho)$$

of 4.2 is isomorphic to a complex of dimension $n-1$ and homotopy equivalent to a wedge of $(n-1)$-spheres if $q \geq 2^{\dim(L_{i+1}/L_i)-2}$ for $i = 0, \ldots, s$, hence if $q \geq 2^{n-1}$, the worst case being $\dim\sigma = 0, L_1 = \pi^{-1}L_0$. Recall that $\dim V = n+1$ in 4.2. ∎

5.1.4. Remark: Our definitions of degree and canonical filtration are special cases of the usual ones ([24, 16, 19]). More precisely, let L be an R-lattice in $V = K^{n+1}$. In the constant sheaf on $P^1(\bar{k})$ with stalk V let $E(L)$ be the subsheaf whose stalks are $E(L)_P = \oplus R_P\, e_i$ for $P \in \bar{k}$ and $E(L)_\infty = L$, resp. Then $E(L)$ is the sheaf of germs of sections of a vector bundle, also denoted $E(L)$. Then for $x \in V, x \neq 0, \deg^L(x)$ is the degree of the line bundle $E(L) \cap$ span x. The canonical filtration of the vector bundle $E(L)$ is $\{E(L) \cap U | U \in \tau(L)\}$.

Then $\tau(L')$ induces the flag $\tau(L')_{\pi^{-1}L'/L'}$ in the fibre $\pi^{-1}L'/L'$ of $E(L')$ over ∞. Proposition 5.1.2 describes the chambers $C \supset \sigma$ of minimal distance from ΓC_0 in terms of certain subspaces of the fibres of $E(L_i)$ at $\infty, [L_i] \in \sigma$. ∎

5.2 Stacking Boxes. To prove Proposition 5.1.1 we first restate it.

Let $m \leq m'' \leq m+e$ be in $\mathbb{Z}^{n+1}_{\text{mon}}$. We say that m' is the **successor** of m under m'' if $m' \in \mathbb{Z}^{n+1}_{\text{mon}}, m' = m + e_j \leq m''$ and $m_j = \min\{m_i | m_i < m_i''\}$. Think of m as $n+1$ stacks of boxes, the stack number i consisting of m_i boxes, and of m'' as a ceiling under which new boxes can be stacked. Then $m \leq m'' \leq m+e$ means that on stack number i there fit $\varepsilon_i = m_i'' - m_i \in \{0,1\}$ boxes. The conditions $m, m', m'' \in \mathbb{Z}^{n+1}_{\text{mon}}$ say that the height of the stacks decreases from left to right. So to perform the move from m to its successor m' under m'' means to stack one new box on the lowest possible level and to put it on this level as far left as possible, since $m' \in \mathbb{Z}^{n+1}_{\text{mon}}$.

There is a successor of m under m'' iff $m < m''$, in which case it is unique. A sequence $m = m_0 < \cdots < m_t = m''$ is called the **sequence of successors** from m to m'' if m_i is the successor of m_{i-1} under m'' for every $i = 1, \ldots, t$.

The canonical filtration $\tau(L(m))$ for $L(m)$ has as its first non-zero term V_1 the span of those e_i, for which m_i is maximal, as V_2 the span of those e_i, for which m_i is maximal or second maximal, etc. (see Fig. 1).

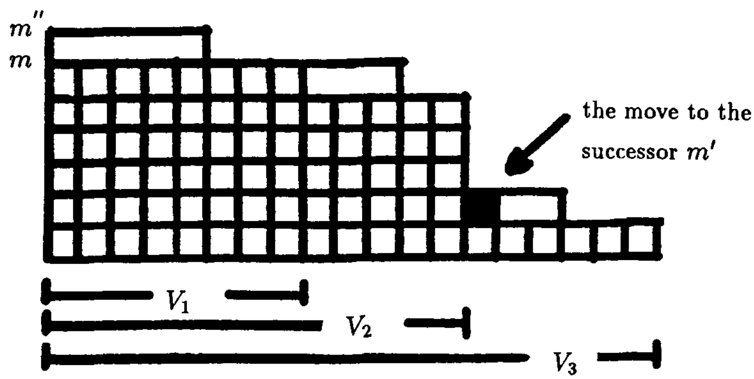

Fig.1.

The following lemma follows immediately from the definitions and the computation of the canonical filtration of $L(m)$ done at the beginning of 5.1.

5.2.1 LEMMA: *Given $L' = L(m') < L'' = L(M'') \leq \pi^{-1}L'$. Let*

$$L' = L(m') = L(m_0) < L(m_1) < \cdots < L(m_t) = L(m'') = L'', \qquad m_j \in \mathbf{Z}^{n+1}_{\text{mon}},$$

be a maximal ascending sequence from L' to L''. Then $L(m_j)/L(m_0)$ is transversal to the flag $\tau(L')_{L''/L'}$ for every $j = 0, \dots, r$ iff $m' = m_0 < m_1 < \cdots < m_t = m''$ is the sequence of successors of m' under m''.

So 5.1.1 is equivalent to

5.2.2 PROPOSITION: *Let $\sigma = \{[L_0], [L_1], \dots, [L_s]\}, L_i = L(m_i)$, be a simplex of $F, m_0 < m_1 < \cdots < m_s < m_0 + e =: m_{s+1}$. Let C be a chamber containing σ. Then $d(C_0, \sigma) = d(C_0, C)$ iff $C \in F$ — equivalently every vertex of C is of the form $\Lambda(m), m \in \mathbf{Z}^{n+1}_{\text{mon}}$ — and $\{m \in \mathbf{Z}^{n+1}_{\text{mon}} | \Lambda(m) \in C, m_i \leq m \leq m_{i+1}\}$ is the sequence of successors of m_i under m_{i+1} for every $i = 0, \dots, s$.*

Proof: Let σ be an arbitrary simplex of a building Δ and C_0 a chamber of Δ. Then there is a unique chamber $C \supset \sigma$ of Δ such that $d(C_0, C) = d(C_0, \sigma)$ (the projection of C_0 on σ of [29] 2.30, cf. [15] IV. 6 Lemma 1). It thus suffices to prove necessity in the proposition, since the successor properties of the proposition determine C uniquely.

If C is the chamber such that $d(C_0,\sigma) = d(C_0, C)$, then c is contained in every root containing σ and C_0 ([29] Theorem 2.19). Recall that a root is a halfspace in an apartment whose boundary is a wall. It follows that in our case $C \in F$ since F is an intersection of roots containing C_0 and σ. Furthermore, if C is not as described in the proposition then there is a $\Lambda(m_i) \in \sigma$ and a $m_t \in \mathbb{Z}_{\text{mon}}, \Lambda(m_t) \in C$ such that m_t is not the successor of m_{t-1} under m_i but $m_{t+1}, \cdots, m_i$ is the sequence of successors of m_t under m_i, say

$$m_t = m_{t-1} + e_j,$$
$$m_{t+1} = m_t + e_k,$$

$m_{t-1,j} = d, m_{t,k} = d'$ with $d > d'$. Then $j < k$. Let ζ be the linear form on $\mathbb{R}^{n+1}, \zeta(x_0,\ldots,x_n) = x_j - x_k$. Then both σ and C_0 are in the halfspace $\zeta(x) \le d - d'$ bounded by the wall $\zeta(x) = d - d'$, hence so is C, a contradiction.

5.2.2 makes it easy to compute $\mathcal{R}(C)$ (see 2.1).

5.2.3 COROLLARY: *Let C be a chamber in F, consisting of the vertices $\Lambda(m_i)$, $m_0 < m_1 < \cdots < m_{n+1} = m_0 + e$, $m_i \in \mathbb{Z}^{n+1}_{\text{mon}}$. Then $\Lambda(m_t) \in \mathcal{R}(C)$ iff m_t is not the successor of m_{t-1} under m_{t+1}.*

Here $t = 0, \cdots, n$, put $m_{-1} = m_n - e$. ∎

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